Reflections on a result by Domingo Gomez

Haakon Waadeland

Norwegian University of Science and Technology, Trondheim

1. Introduction

The following result, presented on page 184 in (1) goes back to fo Domingo Gomez [2], [3] .

The number $2+2^{1/3}$ can be represented by a branched continued fraction: The equation

$$C = 3 + \frac{1}{3} \underset{+}{} \frac{C}{3} \underset{+}{} \frac{C}{3} \underset{+}{} \frac{C}{3} \underset{+}{} \cdots, \tag{1}$$

has $C = 2+2^{1/3}$ as a solution. In (1) we replace on the righthand side repeatedly C by the same continued fraction, then

$$2+2^{1/3} = C = 3 + \frac{1}{3} \underset{+}{} \frac{3+\frac{1}{3} \underset{+}{} \frac{C}{3} \underset{+}{} \frac{C}{3} \underset{+}{} \cdots}{3} \underset{+}{} \frac{3+\frac{1}{3} \underset{+}{} \frac{C}{3} \underset{+}{} \frac{C}{3} \underset{+}{} \cdots}{3} \underset{+}{} \cdots.$$

The first approximants are

$$C_0 = 3, \quad C_1 = 3 + \frac{1}{3}, \quad C_2 := 3 + \frac{1}{3} \underset{+}{} \frac{C_0}{3},$$

$$C_n := 3 + \frac{1}{3} \underset{+}{} \frac{C_{n-2}}{3} \underset{+}{} \cdots \underset{+}{} \frac{C_0}{3} = 3 + \frac{1}{3 + C_{n-2}(C_{n-1}-3)}.$$

Already the approximants of order 4, 5, 6 are very good, se [1], page 184.

2. Why $a = 3$?

A natural (and vague) question here is: Let a be a positive number. Is in

$$C = a + \frac{1}{a} \underset{+}{} \frac{C}{a} \underset{+}{} \frac{C}{a} \underset{+}{} \frac{C}{a} \underset{+}{} \cdots \tag{2}$$

$a = 3$ the only value for which we get such a nice result? In case of YES, then WHY? An attempt to come up with a possible answer starts with the observation that the continued fraction (2) converges for all C not on the ray $(-\infty, -a^2/4)$. In the following we assume this to be the case. Actually, we are looking for *positive* solutions C. Let U be the value of the continued fraction

$$U := \frac{C}{a\ +}\ \frac{C}{a\ +}\ \frac{C}{a\ +}\ \cdots . \tag{3}$$

Then we have

$$U := \frac{C}{a+U},$$

leading to

$$U := -\frac{a}{2} + \sqrt{C + \frac{a^2}{4}},$$

and hence, from (2)

$$C := a + \frac{1}{\frac{a}{2} + \sqrt{C + \frac{a^2}{4}}}.$$

Observe that any solution C has to be $\geq a$. A rearrangement followed by a squaring gives

$$(\frac{1}{C-a} - \frac{a}{2})^2 = C + \frac{a^2}{4},$$

and finally

$$\frac{1}{(C-a)^2} - \frac{a}{C-a} - C = 0.$$

For $C \neq a$ we get the cubic equation

$$H := x^3 - 2ax^2 + (a + a^2)x - 1 - a^2 = 0, \tag{4}$$

where we have replaced C by x . The solution is well known from the time of the renaissance (Cardano and others). We let MAPLE do the work for us and describe the result as follows:

$$T := \left(36a^2 - 8a^3 + 108 + 12\sqrt{-3a^4 + 54a^2 + 81}\right)^{1/3}, \quad U := \left(\frac{1}{3}a - \frac{1}{9}a^2\right). \tag{5}$$

The roots are:

$$x_1 = \frac{T}{6} - \frac{6U}{T} + \frac{2a}{3}, \quad x_2 = \frac{-T}{12} + \frac{3U}{T} + \frac{2a}{3} + \frac{\sqrt{3}}{2} I \left(\frac{T}{6} + \frac{6U}{T}\right);$$

$$x_3 = \frac{-T}{12} + \frac{3U}{T} + \frac{2a}{3} - \frac{\sqrt{3}}{2} I\left(\frac{T}{6} + \frac{6U}{T}\right);$$

For $a = 3$ we have $U = 0$, and the roots are very simple. We have

$$T := 6 \cdot 2^{(1/3)},$$

and hence the roots are

$$2^{(1/3)} + 2, \quad e^{(2I\pi/3)} \cdot 2^{(1/3)} + 2, \quad e^{(4I\pi/3)} \cdot 2^{(1/3)} + 2.$$

Only the first one is of interest to us.

This gives an answer to the two questions raised. The two additional solutions are solutions of the equation after squaring, but not of our problem.

By going to other types of continued fractions similar questions may be studied.

3. Another example.

For a cubic equation with real coefficients like the equation (4) there are different cases to study: one real root and two complex conjugate roots, three real roots, where in special cases two or three may coincide.

The positive solutions $> a$ of (4) are possible C-values in (2), but by far not as nice as in the case by Domingo Gomez. We are not going into a discussion of the different cases. We choose as an example the case where the factor of I is 0, i.e. when

$$\frac{T}{6} + \frac{6U}{T} = 0,$$

or

$$Q := T^2 + 36U = 0,$$

or, in terms of a:

$$Q := \left(36a^2 - 8a^3 + 108 + 12\sqrt{-3a^4 + 54a^2 + 81}\right)^{(2/3)} + 12a - 4a^2.$$

The solutions R of the equation $Q = 0$ give the a-values for which $x_2 = x_3$. We find

$$R := \sqrt{9 + 6\sqrt{3}}, -\sqrt{9 + 6\sqrt{3}},$$

and stick to the positive value

$$R = \sqrt{9 + 6\sqrt{3}} = 4.403669476 \tag{6}$$

in the following.

We find

$$x_1 = \frac{T}{6} - \frac{6U}{T} + \frac{2a}{3} = 1 + \sqrt{3} + (1 - \frac{\sqrt{3}}{3})\sqrt{9 + 6\sqrt{3}} \quad = \quad 4.593260526 \tag{7}$$

and

$$x_2 = x_3 = \frac{-T}{12} - \frac{3U}{T} + \frac{2a}{3} = -\frac{1}{2}(1+\sqrt{3}) + \frac{1}{2}(1+\frac{\sqrt{3}}{3})\sqrt{9 + 6\sqrt{3}} \quad = \quad 2.107039213 \tag{8}$$

The equation (4) is thus satisfied with $a = R$ from (6) and $C = x_1$ from (7) or $C = x_2 = x_3$ from (8). Only the first one is a solution of (2):

$$C = a + \frac{1}{a\,+}\,\frac{C}{a\,+}\,\frac{C}{a\,+}\,\frac{C}{a\,+}\,\cdots \tag{2}$$

with $a = \sqrt{9 + 6\sqrt{3}}$ and $C = 1 + \sqrt{3} + (1 - \frac{\sqrt{3}}{3})\sqrt{9 + 6\sqrt{3}}$.

References

1. A.Cuyt, V.Brevik Petersen, B.Verdonk, H.Waadeland, W.B.Jones, *Handbook of Continued Fractions for Special Functions*, Springer, 2008.

2. Steven R. Finch, *Mathematical Constants, p. 3-4*, vol 94 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, 2003.

3. Domingo Gomez Morin, *La Quinta Operacion Aritmetica,* ISBN: 980-12-1671-9